Cosmological model with Ω_M-dependent cosmological constant

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Abstract

The idea here is to set the cosmical constant λ proportional to the scalar of the stress-energy tensor of the ordinary matter. We investigate the evolution of the scale factor in a cosmological model in which the cosmological constant is proportional to the scalar of the stress-energy tensor.

1 Introduction

The observational view of the universe has drastically changed during the last ten years. New observation suggests a universe that is leigh-weight, is accelerating, and is flat [10] [1] [6]. One way to account for cosmic acceleration is the introduction a new type of energy, the so-called *quintessence* ("dark energy"), a dynamical, spatially inhomogeneous form of energy with negative pressure [12]. A common example is the energy of a slowly evolving scalar field with positive potential energy, similar to the inflation field in the inflation cosmology. The quintessence cosmological scenario (QCDM) is a spatially flat FRW space- time dominated by the radiation at early time, and cold dark matter (CDM) and quintessence (Q) at later time. A series of papers of Steinhardt et al. is devoted to the various quintessence cosmological models [11] (a number of follow-up studies are underway). The quintessence is supposed to obey an equation of state of the form

$$p_Q c^{-2} = w_Q \varrho_Q, \qquad -1 < w_Q < 0. \tag{1}$$

In many models w_Q can vary over time. For the vacuum energy (static cosmological constant), it holds $w_Q = -1$ and $\dot{w}_Q = 0$.

In what follows we present a variant of the quintessence cosmological scenario in which the content of black energy is given by the cosmological constant. The possible

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existence of very small but non-zero cosmological constant revives in these days due to new observation in cosmology. In the absence of a symmetry in nature which would set the value of λ to precisely zero, one is forced to either set $\lambda \neq 0$ by hand, or else look for mechanisms that can generate $\lambda = \lambda_{obs} > 0$, where $\Lambda \approx 10^{29} gcm^{-3}$ is the value of the Λ-term inferred from recent supernovae observation. There are several mechanism which could, in principle, give rise either to time independent constant, or else a time dependent Λ-term. Models with a fixed Λ run into fine-tuning problems since the ratio of the energy density in Λ to that of matter/radiation must be tuned to better than one part in 10^{60} during the early universe in order that $\Lambda \approx \rho_{matter}$ today. Scalar field models considerably alleviate this problem though some fine-tuning does remain in determining the 'correct choose' of parameters in the scalar field potential.

Due to this fact, there are many phenomenically ansatzes for the cosmological constant, e.g. the different built-in cosmological constants (for a detailed analysis of these models see [22]) which are more or less justified by the physical arguments. We remark that observational data indicate that $\lambda \approx 10^{-55} cm^{-2}$ while particle physics prediction for λ is greater than this value by factor of order 10^{120}. This discrepancy is known as the *cosmological constant problem*. The vacuum energy assigned to λ appears very tiny but not zero. However, there is no really compelling dynamical explanation for the smallness of the vacuum energy at the moment [2] (simple quantum-mechanical calculations yield the vacuum energy much larger [4]). The quintessence eventually modelled by a positive non-zero cosmological constant helps overcome the age problem, connected on the one side with the hight estimates of the Hubble parameter and with the age of globular clusters on the other side. To explain this apparent discrepancy the point of view has often been adopted which allows the cosmological constant to vary in time. The idea is that during the evolution of universe the "black" energy linked with cosmological constant decays into the particles causing its decrease.

It is well-known that the Einstein field equations with a non-zero λ can be rearranged so that their right-hand sides consist of two terms: the stress-energy tensor of the ordinary matter and an additional tensor

$$T_{ij}^{(\nu)} = \left(\frac{c^4 \lambda}{8\pi G} \right) g_{ij} = \Lambda g_{ij}. \tag{2}$$

In common discussions, Λ is identified with vacuum energy because this quantity satisfies the requirements asked from Λ, i.e. (i) it should have the dimension of energy density, and (ii) it should be invariant under Lorentz transformation. The second property is not satisfied for arbitrary systems, e.g. material systems and radiation. Gliner [3] has shown that the energy density of vacuum represents a scalar function of the four-dimensional space-time coordinates so that it satisfies both above requirements. This is why Λ is commonly *identified* with the vacuum energy.

However, there may be generally other quantities satisfying also the above requirements. Instead of identifying Λ with the vacuum energy we have identified Λ in [13] with the stress-energy scalar $T = T_i^i$ a scalar which arises by the contraction of the stress-energy tensor of the ordinary matter T_i^j. This quantity *likewise* satisfies both above requirements, i.e., it is Lorentz invariant and has the dimension of the energy density. Hence, we make the ansatz

$$\Lambda_A = \frac{c^4 \lambda_A}{8\pi G} = \kappa T_i^i = \kappa T \tag{3}$$

or

$$\lambda_A = \frac{8\pi G \kappa T}{c^4}, \tag{4}$$

where κ is a dimensionless constant to be determined. Λ_A is a dynamical quantity, changing over time, representing, in the quintessence theory, the quintessence component. In contrast with some other cosmological models, we suppose that the universe consists of a mixture of the *ordinary* mass-energy and the quintessence component functionally linked with T via the cosmological constant λ_A. We note that there are similar attempts to identify λ with the Ricci scalar (see [21]).

We describe a cosmological model in which we consider (3) as a phenomenological ansatz for the cosmological constant. Assuming the flatness of space the constant κ is uniquely given. This model of the universe we confront with the observation and find that it is in concord with the data. The word "phenomenically" means that no attempt to derive these models from the underlying quantum field theory is being made. Historically, an array of the phenomenological Λ-models were proposed since 1986. These model may be classified into two groups: (i) kinetical models where Λ is simple assumed to be function of either the cosmological time t, the scale factor $a(t)$, etc., of the FRW cosmological model and (ii) field-theoretical models. Here the Λ-term is assumed to be new physical classical field with some phenomenological Hamiltonian. The phenomenological model introduced here does not belong to any of these classes since T is not a kinetical quantity; rather it can be considered as a model with possible field-theoretical background.

2 Friedmann's model with a Ω_M-dependent cosmological constant

The standard Einstein field equations (see, e.g. [24]) can be written in the form

$$R_{ij} - g_{ij}(1/2)R = \frac{8\pi G}{c^4}(T_{ij}^{(m)} + T_{ij}^{(v)}), \tag{5}$$

where $T_{ij}^{(m)}$ is the energy-momentum tensor for the perfect fluid [23]

$$T_{ij}^{(m)} = (\rho + p/c^2)u_i u_j - p g_{ij} \tag{6}$$

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and

$$T_{ij}^{(v)} = g_{ij}\Lambda \quad \Lambda = \frac{\lambda c^4}{8\pi G}. \tag{5a}$$

Putting $\Lambda = \Lambda_A = \kappa T$ we have $T_{ij}^{(v)} = g_{ij}\kappa T$. Inserting Eqs.(5a) and (5) into Eq.(5) we have

$$R_{ij} - g_{ij}(1/2)R = \frac{8\pi G}{c^4}\left[(\rho + p/c^2)u_i u_j - (p - \kappa T)g_{ij}\right]. \tag{7}$$

The stress-energy tensor of the cosmic medium T_j^i in the everywhere local rest frame has only four non-zero components $T_0^0 = \varrho c^2, T_1^1 = T_2^2 = T_3^3 = -p$ [5]. Therefore,

$$T = \varrho c^2 - 3p. \tag{8}$$

Inserting Eq.(8) into Eq.(7) we get

$$(T_{ij}^{(m)} + T_{ij}^{(v)}) = (\rho + p/c^2)u_i u_j - [p(1 + 3\kappa) - \kappa\rho c^2]g_{ij}, \tag{9}$$

This is the energy-momentum tensor for a perfect fluid with effective density $\hat{\rho}$ and pressure $\hat{p}$.

$$T_{ij}^{(m)} + T_{ij}^{(v)} = (\hat{\rho} + \hat{p}/c^2)u_i u_j - \hat{p}g_{ij}. \tag{10}$$

The quantities $\hat{\rho}$ and $\hat{p}$ can be determined given the equation of state $p = w\rho c^2$.

Our next main concern will be to find the evolution of scale factor for Friedmann's equation (7) in the radiation-dominated and matter-dominated eras. In a homogeneous and isotropic universe characterized by the Friedmann-Robertson-Walker line element the Einstein equations with matter in the form of a perfect fluid and non-zero cosmical term λ acquire the following forms

$$3\frac{\dot{R}(t)^2}{R(t)^2} = 8\pi G\rho + \lambda c^2 - 3\frac{kc^2}{R^2(t)}, \tag{11}$$

and

$$\ddot{R}(t) = \frac{4\pi G}{3}(-\rho - 3p/c^2) + \frac{\lambda c^2}{3}R(t), \tag{12}$$

where $R(t)$ is the time-dependent scale factor.

A quantitative analysis of solutions to Eqs.(11) and (12) can be gained by eliminating ρ in these equations and combining them into a single equation for the evolution of the scale factor in the presence of a λ-term [29]

$$\frac{2\ddot{R}}{R} + (1 + 3w)(\frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2}) - (1 + w)\lambda c^2 = 0, \tag{13}$$

Here, we set the equation of state in form $p = \omega\varrho c^2$.

To determine Λ_A which is to be inserted in Eqs.(8) and (9) we have to specify κ. The dimensionless constant κ we determine by assuming that the universe is flat, i.e., $\Omega_{tot} = 1$ which is consistent with the inflationary cosmology density ($\Omega_{tot} = 1$) and conformed by

the measurement of the cosmic microwave background anisotropy [18]. Since $\Omega_M < 1$ we suppose that the remaining energy of cosmological constant required to produce a geometrical flat universe is given by the equation

$$\Omega_M + \Omega_\Lambda = \Omega_M + \kappa\Omega_M = 1.$$

This gives

$$\kappa = \frac{1}{\Omega_M} - 1. \tag{10}$$

Inserting Eq.(10) into Eq.(3) we get

$$\Lambda_A = \left(\frac{1}{\Omega_M} - 1\right) T = \left(\frac{1}{\Omega_M} - 1\right)(\rho c^2 - 3p). \tag{14}$$

In the radiation-dominated $p = \rho c^2/3$, hence $\Lambda_M = 0$, i.e. the cosmological evolution in this era is described by the standard FRW model with zero cosmological constant. All processes which took place during this era (e.g., nucleosynthesis etc.) are described by the standard model. However, in the matter-dominated era with $p = 0$ and we get the following effective density and pressure

$$\hat{\rho} = \rho + \kappa\rho, \qquad \hat{p} = -\kappa\rho c^2.$$

Since in the radiation-dominated era is described by standard model we will not further deal with it, instead we will investigation the evolution of the scale factor in dependence on Ω in the matter-dominated era.

3   Matter-dominated epoch

While in the pressure-dominated universe the effect of the cosmological constant on the evolution of the scale factor is zero, in the matter-dominated era it affects this evolution considerably. Inserting $T = \rho_M c^2$ into the equation for Λ_A yields

$$\Lambda_A = \kappa\varrho_M c^2 = \left(\frac{1}{\Omega_M} - 1\right)\rho_M c^2 = (\varrho_{crit} - \varrho_M.)c^2 = \varrho_{crit}(1 - \Omega_M)c^2, \tag{15}$$

We obtain the critical density ρ_{crit} by means of Eqs.(15) and Eq.(11)

$$8\pi G\rho_{crit} = \frac{3\dot{R}^2}{R^2}. \tag{16}$$

Inserting Eq.(16) into Eq.(14) we get immediately the equation for the evolution of $R(t)$ for the matter dominated era

$$\ddot{R}(t) = \left(1 - \frac{3}{2}\Omega_M(t)\right)\frac{(\dot{R}(t))^2}{R(t)}. \tag{17}$$

Its exact solution can be found for an arbitrary time function $\Omega_M(t)$. With the ansatz $R = \exp(y)$, we have

$$\dot{R} = \dot{y}\exp(y), \qquad \ddot{R} = (\ddot{y} + (\dot{y})^2)\exp(y)$$

which inserting into Eq.(17) yields

$$-(2/3)\Omega_M(t)(\dot{y})^2 = \ddot{y}.$$

By putting $\dot{y} = q$, this equation becomes the form

$$-(2/3)\Omega_M(t) = \frac{\dot{q}}{q^2},$$

the solution to which is

$$q = \frac{1}{\int(2/3)\Omega_M(t)dt + C_1}.$$

Since $\dot{y} = q$ we have

$$y = \int\left(\frac{1}{\int(2/3)\Omega_M(t)dt + C_1}\right)dt + C_2.$$

With $y(t)$, the general solution of Eq.(17) is

$$R(t) = \exp\int\left(\frac{1}{\int(2/3)\Omega_M(t)dt + C_1}\right)dt + C_2, \tag{18}$$

where C_1 and C_2 are the integration constants.

In what follows we assume that Ω_M does not change during the matter-dominated era, therefore, the solution of Eq.(18) is

$$R(t) = \left(\frac{3}{2}\right)^{\frac{2}{3\Omega_M}} (\Omega_M C_1 t + \Omega_M C_1 C_2)^{\frac{2}{3\Omega_M}}. \tag{19}$$

To go further we have to specify Ω_M and the boundary conditions for the differential equation (18). For Ω_M we take its observable value. There is growing observational evidence that the total matter of the universe is significantly less than the critical density. Several authors [7] [8] [9] have found that the best and simplest fit is provide by ($h = 0.65 \pm 0.15$)

$$\Omega_M = \Omega_{CDM} + \Omega_{baryon} \approx [0.30 \pm 0.10] + [0,04 + \pm 0.01].$$

As the boundary condition we set $R(t = 0) = 0$. Inserting $\Omega_M = 1/3$ into Eq.(19) and respecting the the previous boundary condition, the evolution factor $R(t)$ take the form

$$R(t)_{(\Omega_M = 1/3)} = \frac{C_1 t^2}{4}. \tag{20}$$

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The time-dependence of the scale factor (20) implies a model of the universe with the following properties:

(i) The Hubble parameter $H = \dot{R}/R = 2/t$, i.e. the age of this universe t_0 is approximately 2.10^{10} yr. In the cosmological model with Λ_A, the universe is old enough for the evolution of globular clusters.

(ii) The decelerator q_0 is an important parameter of any model of the universe. It probes the equation of state of matter and the cosmological density parameter. In our model, it takes the value $q_0 = -1/2$, i.e. the universe is accelerated in concord with the recent observation.

(iv) The cosmical constant Λ_A is time-dependent $\lambda_A = 8/(c^2 t^2)$. It is interesting that $\lambda \propto t^{-2}$ was phenomenologicaly set by several authors [18]-[22].

(v) In the considered model the universe is causally connected. The proper distance $L(t)$ to the horizon, which is the linear extent of the causally connected domain, diverges

$$L(t) = R(t) \int_0^t \frac{d\tau}{R(\tau)} = C_2 t^2 [-(\frac{2}{\tau})|_0^t] = -\infty,$$

In [25] is shown that the only way to make the whole of the observable universe causally connected is to have a model with infinite $L(t)$ for all $t > 0$, i.e. in our model the whole observable universe is causally connected. We remark that for $\Omega_M = 1$, $R(t) \propto t^{2/3}$, i.e. the evolution law of $R(t)$ is in a pressure-free medium, identical with that of Standard Cosmology.

As Ω_M decreases, $R(t)$ passes smoothly to the form

$$R(t) = \exp(C_1(t - C_2)). \tag{21}$$

It is tempting to choose for the early universe $\Omega_M = 0$, i.e. to suppose that the universe started in a massless state and its mass content was created later through the decay of the cosmical term. Under this assumption we have

$$R(t) = \exp(C_1(t - C_2)) = R_0 \exp(C_1 t), \qquad C_1 = \frac{1}{t_0}. \tag{19}$$

The natural measures for length and time in cosmology is the Planck length and time, i.e., $l_p = (Gh/c^3)^{1/2} = 4.3.10^{-35}$m and $t_p = (Gh/c^5)^{1/2} = 1.34.10^{-43}$s, respectively. It seem to be reasonable to assume that at the very beginning of the cosmic evolution the radius of the universe was of the order of the Planck length, therefore we put in Eq.(19) the integration constant R_0 and C_1 equal to l_p and $1/t_p$, respectively. Then, we get for the initial radius and the velocity the values

$$R(0) = l_p = 4.3.10^{-35} m. \qquad \dot{R}(0) = \frac{l_p}{t_p} = c = 3.10^8 ms^{-1},$$

respectively. The most interesting feature of this universe is its inflationary character.

In order to vanish the covariant divergence of the right-hand side of Eq.(6) the matter is created along with energy and momentum. Therefore, the cosmological constant λ_A is decaying and transforming its energy into particles and/or radiation. Observationally, such an effect can, in principle, be tested: in the case of dissipative, baryon number conserving decay of a Λ-term into baryons and antibaryons, the subsequent annihilation of matter and antimatter would result in a homogeneous gamma-ray background in the universe [26]. A decay of the cosmological term directly into radiation could be probed by the microwave background anisotropies and the cosmological nucleosyntethis. Supposing that the cosmological constant is decaying via particles, the present rate of the particle creation (annihilation)

$$ n = \frac{1}{R_0^3} \frac{d(\rho R^3)}{dt} |_0, $$

(where the subscript '0' denotes the present value of the corresponding quality) in the considered model is $2\rho H$ less than in Steady state cosmology ($3\rho H$). We remark that the free energy of the decaying λ may cause also other effects than the creation of particles (nucleons) or radiation. It can be stored, e.g. in form of small vacuum excitations of the gravitation field (see [27]). The detailed discussion of this topic would exceed the scope of this paper.

4   Final remark

Summing up, we can state:

(i) In previous sections we have shown that in the basic dynamical equation (17) the energy density does not explicitly appear only the density parameter Ω_M. We note that the density parameter Ω_M as the ratio of ρ_M and ρ_{crit} may be *finite* although both quantities are infinite.

(ii) In the recently popular ΛCDM cosmological model, which consists of a mixture of vacuum energy and cold dark matter, a serious problem exists called in [11] as the cosmic coincidence problem. Since the vacuum energy density is constant over time and the matter density decreases as the universe expands it appears that their ratio must be set to immense small value ($\approx 10^{-120}$) in the early universe in order for the two densities to nearly coincide today, some billions years later. No coincidence problem exists in our model of the universe because Λ_A here is functionally connected with Ω_M in such a way that this ratio in the matter dominated epoch does not vary over time.

(iii) In the radiation dominated epoch $w = 1/3$ and, according to Eq.(8), $T = 0$. The evolution dynamics in this epoch runs so as if $\lambda = 0$. Let us remark that in the string-dominated universe ($w = 2/3$) the cosmological constant becomes negative!

The relation between w_Q and κ is [13]

$$w_Q = -\frac{\kappa}{1+\kappa}.$$

New measurement required $w_Q \leq -0.7$ [28]. Inserting κ for could dark matter $Q_M \approx 0.3$ we have $w_Q \leq -0.7$. It is noteworthy, that in the limiting case when $Q_M \to 0 \; \kappa \to \infty$, i.e. an almost empty spacetime behaves similar as a space-time with the static cosmological constant. In [13] there are graphs of the angular-diameter distance on redshift for the Friedmann cosmologies with selected value of Q_M and κ.

In conclusion, the cosmological parameters of our cosmological models are comfort with the recent observational data of the flat and acceleration universe. The described universe is leigh-weight, is strictly flat, is accelerating, is old enough and is causally connected. One can speculate about the linear functional dependence of Λ on T. The simplest hypothesis seems to be that T is source of an unknown classical field whose quintessence energy density is proportional to Ω_M.

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